Five Exact Ramsey Numbers for Pairs of Complete Bipartite Graphs

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Abstract

We record five ordinary two-colour Ramsey numbers for complete bipartite graphs:

$$R(K_{2,17}, K_{2,17}) = 66,$$
$$R(K_{2,29}, K_{2,30}) = 116, \qquad R(K_{2,39}, K_{2,40}) = 156,$$
$$R(K_{2,43}, K_{2,44}) = 172, \qquad R(K_{2,47}, K_{2,48}) = 188.$$

The diagonal equality follows by combining the characterization of Exoo, Harborth, and Mengersen with Gritsenko's strongly regular graph with parameters $(65, 32, 15, 16)$. The other four follow from the theorem of Lortz and Mengersen and symmetric Hadamard matrices of orders 116, 156, 172, and 188 constructed in 2015–2018. Consequently,

$$R(K_{2,n-1}, K_{2,n}) = 4n - 4 \qquad (3 \le n \le 58).$$

The April 2026 revision of the dynamic survey *Small Ramsey Numbers* records the diagonal interval as open and identifies these four asymmetric cases as unresolved. We include deterministic checks of all five finite certificates. The contribution is an explicit reconciliation of published conditional Ramsey theorems with later published constructions; no new strongly regular graph or Hadamard matrix is claimed. The resulting exact determinations close five entries in the maintained survey and complete its stated almost-diagonal range through $n = 58$.

1 Introduction

For graphs G and H, let $R(G, H)$ denote the least integer N such that every red-blue colouring of $E(K_N)$ contains a red copy of G or a blue copy of H. Exact values are rare even when both forbidden graphs are complete bipartite graphs with one part of size two.

The current revision of Radziszowski's dynamic survey records

$$65 \le R(K_{2,17}, K_{2,17}) \le 66$$

as the first open diagonal case [13]. The same survey states that, for $3 \le n \le 58$, equality in

$$R(K_{2,n-1}, K_{2,n}) \le 4n - 4$$

is unresolved only for $n = 30, 40, 44, 48$ [13, 9]. Later constructions of strongly regular graphs and Hadamard matrices close all five gaps.

The relevant Ramsey theorems and finite constructions were published separately, and their ordinary complete-host consequences appear not to have been recorded. We make the resulting exact values explicit and give deterministic checks of each finite certificate. The advance is the closure of five named survey gaps and the resulting complete finite-range classification; the mechanism is theorem–object reconciliation rather than a new general Ramsey method, strongly regular graph, or Hadamard matrix.

2 Known Ramsey results

We use the following elementary reformulation throughout.

Lemma 1 (Codegree criterion)**.** *A graph contains a (not necessarily induced) copy of $K_{2,t}$ if and only if some pair of distinct vertices has at least t common neighbours.*

Proof. The two vertices form the part of size two and any t of their common neighbours form the other part. The converse is immediate from a copy of $K_{2,t}$. □

Exoo, Harborth, and Mengersen proved the following characterization [5]:

$$R(K_{2,n}, K_{2,n}) = 4n - 2 \quad \Longleftrightarrow \quad \text{an } \operatorname{srg}(4n-3, 2n-2, n-2, n-1) \text{ exists.} \tag{1}$$

In particular, the general upper bound is $4n - 2$.

Lortz and Mengersen proved the bound [9]:

$$R(K_{2,n-1}, K_{2,n}) \le 4n - 4 \qquad (n \ge 3), \tag{2}$$

with equality whenever a symmetric Hadamard matrix of order $4n - 4$ exists. For completeness, Section 4 includes the lower-bound argument used for the four later matrices.

3 The diagonal value

Theorem 2. $R(K_{2,17}, K_{2,17}) = 66$.

Proof. At $n = 17$, the parameters on the right side of (1) are

$$(4n - 3, 2n - 2, n - 2, n - 1) = (65, 32, 15, 16).$$

Gritsenko constructed a strongly regular graph with exactly these parameters [6]. Colour its edges red and its nonedges blue. Every pair has red codegree 15 or 16. Its complement has parameters

$$(65, 65 - 32 - 1, 65 - 2(32) + 16 - 2, 65 - 2(32) + 15) = (65, 32, 15, 16),$$

so every pair also has blue codegree 15 or 16. Lemma 1 therefore gives a colouring of K_{65} with no monochromatic $K_{2,17}$, and hence $R(K_{2,17}, K_{2,17}) \ge 66$. The upper bound in (1) gives equality. □

Gritsenko prints a complete 65×65 adjacency matrix. The ancillary verifier reads that matrix directly and checks every parameter used below.

4 Four asymmetric values

Lemma 3. *If a symmetric Hadamard matrix of order $4n - 4$ exists, then*

$$R(K_{2,n-1}, K_{2,n}) \ge 4n - 4.$$

Proof. Let H be such a matrix. Negate H if necessary and conjugate it by a diagonal sign matrix so that its first row and column consist entirely of $+1$ entries. Delete that row and column to obtain a symmetric core C of order $4n - 5$. Every row of C sums to -1, and two distinct rows have inner product -1.

For two distinct rows, let a, b, c, d count columns of types $(+,+), (+,-), (-,+), (-,-)$, respectively. The two row sums, the inner product, and $a + b + c + d = 4n - 5$ give

$$a = n - 2, \qquad b = c = n - 1, \qquad d = n - 1.$$

Colour the edge ij of K_{4n-5} positive or negative according to the sign of C_{ij}. Before excluding the endpoint coordinates i and j, any pair has $n - 2$ common positive positions and $n - 1$ common negative positions. Excluding the endpoints can only decrease these counts. By Lemma 1, the positive colour contains no $K_{2,n-1}$ and the negative colour contains no $K_{2,n}$. Thus K_{4n-5} admits the required avoiding colouring. □

Combining Lemma 3 with (2) yields equality.

Theorem 4. *The following four equalities hold:*

$$R(K_{2,29}, K_{2,30}) = 116, \qquad R(K_{2,39}, K_{2,40}) = 156,$$
$$R(K_{2,43}, K_{2,44}) = 172, \qquad R(K_{2,47}, K_{2,48}) = 188.$$

Proof. Di Matteo, Djokovic, and Kotsireas constructed symmetric Hadamard matrices of orders 116 and 172 [4]. Balonin et al. constructed one of order 156 [2], and Balonin, Djokovic, and Karbovskiy constructed one of order 188 [3]. These orders are $4n - 4$ for $n = 30, 40, 44, 48$, respectively. Apply Lemma 3 and the upper bound (2). ☐

Corollary 5. *For every integer n with $3 \leq n \leq 58$,*

$$R(K_{2,n-1}, K_{2,n}) = 4n - 4.$$

Proof. The April 2026 survey states that equality in this range was known except for exactly $n = 30, 40, 44, 48$ [9, 13]. Theorem 4 settles those four remaining cases. ☐

5 Independent finite verification

The accompanying Python verifier uses only the standard library. For the diagonal case it reads the 65 rows printed by Gritsenko and checks the zero diagonal, symmetry, degree 32, and every red and blue pair-codegree. For the four asymmetric cases it reconstructs the propus arrays from the published difference families, verifies all entries, symmetry, and every Hadamard row inner product, normalizes each matrix, and checks every pair-codegree in the derived colouring.

Table 1: Deterministic certificate output. The final two columns give the maximum common-neighbour count in the two colours.

Claim	Witness vertices	First colour	Second colour
$R(K_{2,17}, K_{2,17}) = 66$	65	16	16
$R(K_{2,29}, K_{2,30}) = 116$	115	28	29
$R(K_{2,39}, K_{2,40}) = 156$	155	38	39
$R(K_{2,43}, K_{2,44}) = 172$	171	42	43
$R(K_{2,47}, K_{2,48}) = 188$	187	46	47

The verification files accompanying this article may be checked by running

```
python3 -m loop.complete_bipartite_ramsey
python3 -m unittest tests.test_hadamard_ramsey
```

The verifier, tests, and input certificates are included as ancillary files. The certificate SHA-256 digests are

```
6015bce8548584f108e6b5c51365bd6ae7c46349176f1b0fc87924ec0377df81
  gritsenko-srg65.rows
eef979c2bc99cdd299b963f7af6ba62e169b43b1aa6115cba0488c149d9f1e22
  k2n-ramsey-hadamard.json
```

6 Relation to prior work

The five equalities are consequences of published theorems and published finite constructions. The contribution here is to draw these consequences, record the finite-range corollary, and provide independently checkable certificates. The specialist survey revised on 24 April 2026 records all five positions as unresolved [13].

The derivations are short once the correct sources are placed together, but their consequences change the accepted tabulated record: five stated open intervals become exact values, the first open diagonal case is settled, and the survey's entire almost-diagonal range through $n = 58$ becomes complete. Proof length and record significance are distinct.

As of 10 August 2026, we found no earlier public source stating any of the five equalities. We searched exact and alternative notation, theorem classes, graph parameters, construction orders, forward citations of the construction papers, and the full text of closely related work. General ordinary complete-bipartite bounds and asymptotics do not state these finite conclusions [8, 10, 11, 7]. The closest Hadamard and strongly-regular-graph syntheses concern multipartite host invariants rather than ordinary Ramsey numbers [12, 1]. The ancillary search log records the queries and scope. This search cannot exclude an unindexed or unpublished observation.

Data and code availability. The arXiv ancillary files contain the certificates, deterministic verifier, tests, search log, and SHA-256 manifest. Mathematical verification requires only the Python standard library.

Author contributions and discovery history. Thomas Vergeres developed the Concorde Research System, through which the diagonal consequence was first surfaced. Ian D'Ambrosio led the subsequent Beyond-assisted reconciliation that surfaced the four almost-diagonal consequences and prepared the verifier, prior-art audit, and manuscript. The authors thank the authors and maintainers of the cited surveys, construction papers, and graph databases.

Research-system disclosure. This research was conducted by Ian D'Ambrosio and Thomas Vergeres with assistance from the Concorde and Beyond Research Systems. Beyond, operated by Nth Research Collective, supported literature retrieval, theorem–object matching, certificate reconstruction, code and test development, orchestration, adversarial review, and drafting. The human authors made the final scientific judgments and accept responsibility for the accuracy, integrity, originality, and conclusions. Research systems are not authors.

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