An Exact Counterexample to Affine-Like Price-of-Anarchy Shape in Quartic BPR Routing

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Abstract

We give an exact computer-assisted counterexample to a direct common-degree quartic extension of the affine active-network shape theorem for demand-dependent Price of Anarchy. The instance is a directed network with five vertices, six edges, and three origin–destination paths, all with positive rational costs $c_e(x) = a_e + b_e x^4$. Exact rational interval certificates show that all three paths carry positive Wardrop flow throughout the demand interval $[17, 24]$, while

$$\mathrm{PoA}(21) > \mathrm{PoA}(17), \qquad \mathrm{PoA}(21) > \mathrm{PoA}(24).$$

Continuity therefore forces an interior local maximum despite a constant equilibrium active network. Krawczyk inclusions isolate the Wardrop and social-optimum KKT solutions, elementary boundary inequalities certify constant support, and interval social costs certify both strict comparisons. At differentiability points, we also derive a serial-edge identity explaining how a common quartic edge can alter the derivative of PoA without changing either route split. The proof and certificate use exact rational arithmetic for every interval and sign decision.

1 Introduction

Let a nonatomic population of demand $\mu > 0$ choose routes from one origin to one destination. A Wardrop equilibrium uses only minimum-latency paths. If $C^{\mathrm{eq}}(\mu)$ and $C^{\mathrm{opt}}(\mu)$ are respectively the equilibrium and minimum social costs, the Price of Anarchy is

$$\mathrm{PoA}(\mu) = \frac{C^{\mathrm{eq}}(\mu)}{C^{\mathrm{opt}}(\mu)}.$$

Demand changes can alter both route flows and the set of shortest paths. Cominetti, Dose, and Scarsini formalized the relevant breakpoints using the *active network* $\widehat{\mathcal{E}}(\mu)$, the union of edges on all equilibrium shortest paths, and proved a strong shape theorem for affine costs: between consecutive changes of $\widehat{\mathcal{E}}$, PoA is monotone or has one interior minimum and no interior maximum [4].

That paper also gives a nonlinear counterexample with mixed degrees, $c_1(x) = x$ and $c_2(x) = 1 + x^2$. It does not answer whether the affine shape survives for costs in the standard common-degree quartic BPR form. This distinction matters because common-degree costs obey a scaling relation between equilibrium and the social optimum [7]. Dose's subsequent two-link polynomial analysis concerns demands at which equilibrium is optimal and explicitly leaves polynomial shape results and larger networks open [5]. Related smoothness results require strictly positive edge derivatives and therefore do not apply at configurations containing zero-loaded BPR edges, since $c_e'(0) = 0$ [3].

To our knowledge, the example below is the first exact counterexample to this direct common-degree quartic extension. The counterexample is small, rational, and exactly checkable. Its PoA bump is numerically shallow, but its strictness does not depend on floating-point arithmetic. No claim of topology minimality or empirically calibrated coefficients is made.

2 Model and optimality conditions

For each directed edge e, let

$$c_e(t) = a_e + b_e t^4, \qquad a_e, b_e \in \mathbb{Q}_{>0}.$$

For path flows f and induced edge loads $\ell_e(f)$, Wardrop equilibria are the minimizers of the Beckmann potential [1]

$$\Phi(f) = \sum_e \left(a_e \ell_e(f) + \frac{b_e}{5} \ell_e(f)^5 \right)$$

over the demand simplex. Social optima minimize

$$C(f) = \sum_e \left(a_e \ell_e(f) + b_e \ell_e(f)^5 \right).$$

Thus their path KKT equations use the marginal edge costs $a_e + 5b_e\ell_e^4$. Positivity of the b_e makes both objectives strictly convex in edge loads. In the network below, the path–edge incidence matrix has full column rank, so the path-flow minimizers are unique and continuous in demand.

3 The counterexample

Consider the directed graph in Figure 1. Edge 0 is common to every route, and edges 1–5 form a Wheatstone network. There are exactly three simple origin–destination paths:

$$P_x = (0, 1, 2), \qquad P_y = (0, 3, 4), \qquad P_z = (0, 1, 5, 4).$$

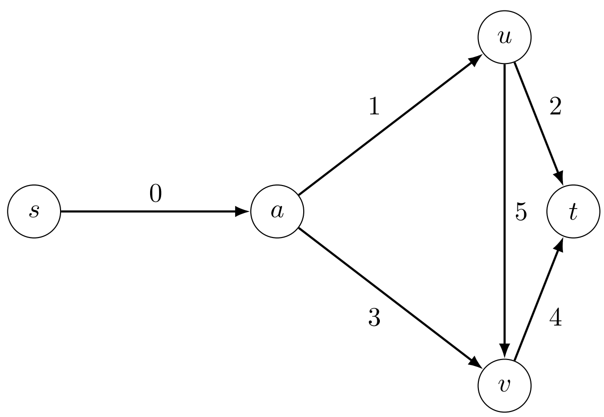

Figure 1: The common-edge Wheatstone network.

The coefficient pairs are given in Table 1.

Table 1: Positive rational coefficients in $c_e(t) = a_e + b_e t^4$.

e	role	a_e	b_e
0	common serial	12,800,000	1855/374
1	upper left	17/3	13/896
2	upper right	236	38/5
3	lower left	1280	17/80
4	lower right	5504/7	188
5	cross edge	424/7	208/3

Theorem 1. *For the network and costs above, all three paths are active at Wardrop equilibrium for every* $\mu \in [17, 24]$. *Nevertheless,* PoA *has an interior maximum in* $(17, 24)$.

3.1 Constant Wardrop support

Let (x, y, z) be the flows on (P_x, P_y, P_z), with $x + y + z = \mu$. Excluding the common edge, the five loads are

$$(x + z, x, y, y + z, z).$$

When all paths are used, equality of P_x and P_z, and of P_y and P_z, is equivalent to

$$F_\mu(x, y) = c_2(x) - c_5(z) - c_4(\mu - x) = 0, \tag{1}$$
$$G_\mu(x, y) = c_3(y) - c_1(\mu - y) - c_5(z) = 0, \tag{2}$$

where $z = \mu - x - y$. Put $\mathbf{H}_\mu = (F_\mu, G_\mu)$. This polynomial map is continuously differentiable. For a rational box X, the verifier evaluates an interval matrix $\mathbf{H}'_\mu(X)$ that contains every Jacobian $\mathbf{H}'_\mu(w)$ for $w \in X$, and applies the two-dimensional Krawczyk operator [6]

$$K(m, X) = m - C\mathbf{H}_\mu(m) + (I - C\mathbf{H}'_\mu(X))(X - m),$$

where m is the midpoint of X and C is the exact inverse of the midpoint Jacobian. The Krawczyk inclusion theorem says that strict inclusion $K(m, X) \subset \operatorname{int} X$ proves a unique root in X. At demand 17 it gives

$$11.745703474 < x < 11.745703476,$$
$$3.439049651 < y < 3.439049654,$$
$$1.815246870 < z < 1.815246875.$$

In particular, the unique equilibrium is in the simplex interior.

It remains to show that this branch cannot meet the boundary before demand 24. The three possible boundary cases admit short exact exclusions.

Lemma 2. *No solution of* (1)–(2) *with nonnegative path flows has* $17 \le \mu \le 24$ *and* $xyz = 0$.

Proof. If $x = 0$, then

$$F_\mu(0, y) \le a_2 - a_5 - a_4 = -\frac{4276}{7} < 0,$$

so (1) cannot hold.

If $y = 0$, equation (2) forces $z < 3$: for $z \ge 3$, $c_1(x + z) + c_5(z) > c_3(0)$. Equation (1) then reads

$$b_2 x^4 = a_5 + a_4 - a_2 + (b_5 + b_4) z^4.$$

Direct substitution at $x = 8, z = 3$ shows that its left side is already larger than the right side. Hence $x < 8$ and $\mu = x + z < 11$.

If $z = 0$, equations (1)–(2) are linear in $X = x^4$ and $Y = y^4$. Their unique solution is

$$X = \frac{3831364840}{18693} > 21^4, \qquad Y = \frac{1083769565}{130851} > 9^4.$$

Thus $\mu = x + y > 30$. □

Lemma 3 (Support continuation)**.** *The unique Wardrop path flow is continuous on* $[17, 24]$. *Starting from the interior root at demand 17, it cannot leave the simplex interior before demand 24.*

Proof. Write a feasible path flow as $f = \mu q$, where q belongs to the fixed compact unit simplex. The Beckmann objective is continuous in (μ, q), and strict convexity together with the full-column-rank incidence matrix makes its minimizer unique. Berge's maximum theorem therefore makes $q(\mu)$, and hence $f(\mu)$, continuous [2].

If a coordinate first reached zero at some $\mu_* \in (17, 24]$, choose demands $\mu_n \uparrow \mu_*$ before that first contact. All three paths are used at μ_n, so (1)–(2) hold there. Continuity of the path flows and costs preserves both equalities in the limit at μ_*. This would give a nonnegative boundary solution forbidden by Lemma 2. □

All three path flows are consequently positive throughout $[17, 24]$, so all three simple paths are shortest. In the active-network convention, $\widehat{\mathcal{E}}(\mu)$ is the full six-edge graph throughout this interval.

3.2 Exact PoA comparisons

The certificate isolates the equilibrium KKT roots at demands 17, 21, and 24. For the social optimum, it isolates an interior marginal-cost KKT root at 17. At 21 and 24 the optimum uses P_x and P_y only; exact univariate brackets for the P_x flow are

$$14.499119464 < u_{21} < 14.499119467, \qquad 16.569846371 < u_{24} < 16.569846374.$$

The marginal cost of the unused path P_z is bounded strictly above the common used-path marginal cost in both boxes.

Substituting these rational flow boxes into

$$C = \sum_{e} (a_e \ell_e + b_e \ell_e^5)$$

gives rational intervals for all six social costs. Dividing outward yields the certified enclosures in Table 2. The displayed decimals are coarser outward roundings of the exact rational endpoints.

Table 2: Exact-rational interval enclosures for PoA.

μ	certified enclosure
17	[1.0000089877, 1.0000089879]
21	[1.0000100959, 1.0000100961]
24	[1.0000088727, 1.0000088730]

The exact checker does not compare rounded decimals. It verifies the positive cross products

$$\underline{C}_{21}^{\mathrm{eq}}\,\underline{C}_{17}^{\mathrm{opt}} - \overline{C}_{17}^{\mathrm{eq}}\,\overline{C}_{21}^{\mathrm{opt}} > 0,$$
$$\underline{C}_{21}^{\mathrm{eq}}\,\underline{C}_{24}^{\mathrm{opt}} - \overline{C}_{24}^{\mathrm{eq}}\,\overline{C}_{21}^{\mathrm{opt}} > 0.$$

Hence PoA(21) strictly exceeds the values at both endpoints. PoA is continuous, so its maximum on $[17, 24]$ is attained in $(17, 24)$. This proves Theorem 1.

4 A serial-edge mechanism

The counterexample was found by exploiting an exact cancellation. Let H be a base network, and fix an open demand interval I on which its equilibrium cost λ and its optimum social cost O are differentiable. Write the equilibrium social cost as $E(\mu) = \mu\lambda(\mu)$ and the optimum marginal cost as $M(\mu) = O'(\mu)$. Add one edge common to every path, with latency $A + B\mu^4$. Its social-cost contribution is

$$G(\mu) = A\mu + B\mu^5.$$

Because every feasible route crosses the new edge, neither the equilibrium nor the optimum route split changes.

Proposition 4. *For every $\mu \in I$, let $N_0 = E'O - EO'$ be the numerator of the derivative of E/O. After adding the common edge, the new derivative numerator is*

$$N = N_0 + A\alpha + B\beta,$$

where

$$\alpha = \mu^2\lambda' + O - \mu M, \qquad \beta = \mu^4\big(\alpha + 4(O - \mu\lambda)\big).$$

Proof. Expand

$$(E' + G')(O + G) - (E + G)(O' + G').$$

The quadratic terms $G'G - GG'$ cancel. Collecting the coefficients of A and B, then using $E = \mu\lambda$, gives the stated formulas. □

At two differentiability points, requiring $N > 0$ at the first and $N < 0$ at the second is therefore a two-variable linear-feasibility problem in positive (A, B). We used this observation as a numerical discovery heuristic to tune a Wheatstone base and common edge; the discovery trajectory is not part of the proof. The common edge leaves its Braess-type path migration untouched while reweighting the derivative through A and B. This explains why untargeted coefficient sampling can miss the effect: the route split and the PoA normalization must be tuned separately.

5 Reproducibility and scope

The archival supplement contains the paper source, canonical certificate, standalone exact checker, SHA-256 manifest, and prior-art record, following standard reproducibility practice [8]. The exact version 1.0.0 archive is permanently available at doi:10.5281/zenodo.21864490. After extracting the supplement, run

```
python3 verify_audit.py
```

This one command checks every manifest entry, runs `ancillary/verify_counterexample.py` in a fresh subprocess, and compares its output byte-for-byte with `ancillary/certificate.json`. The checker uses Python's `Fraction` type for every interval and sign decision; floating-point values appear only as convenience fields after the corresponding exact signs have been checked. The certificate's expected SHA-256 digest is

`1234e8feb90a19dd7dfd29f82cf92af5cd32ec4141e82d1a4f4d4843c05b7eb6.`

Theorem 1 disproves the direct common-degree quartic extension described in the introduction. It does not determine the shape for parallel or series-parallel quartic networks, bound the possible number of extrema, or establish that the topology or coefficient sizes are minimal. The large engineered common free-flow term makes the certified PoA variation small; it does not affect the exact sign comparisons.

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